# Non-degenerate surfaces of revolution in Minkowski space that satisfy the relation $aH + bK = c$

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**Abstract**

In this work, we study spacelike and timelike surfaces of revolution in Minkowski space $\mathbf{E}_1^3$ that satisfy $aH + bK = c$, where $H$ and $K$ denote the mean curvature and the Gauss curvature of the surface and $a$, $b$ and $c$ are constants. The classification depends on the causal character of the axis of revolution and in all the cases, we obtain a first integral of the equation of the generating curve of the surface.

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# 1 Introduction

Consider the three-dimensional Minkowski space $\mathbf{E}_1^3$, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle,\rangle = (dx)^2 + (dy)^2 - (dz)^2$, where $(x, y, z)$ stand for the usual coordinates of $\mathbb{R}^3$. A vector $v \in \mathbf{E}_1^3$ is said spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. A submanifold $S \subset \mathbf{E}_1^3$ is said spacelike, timelike or lightlike if the induced metric on $S$ is a Riemannian metric (positive definite), a Lorentzian metric (a metric of index 1) or a degenerated metric, respectively. In the case that $S$ is a straight-line $L =< v >$, this means that $v$ is spacelike, timelike or lightlike, respectively. If $S$ is a plane $P$, this is equivalent that any orthogonal vector to $P$ is timelike, spacelike or lightlike respectively.

An immersion $x : M \to \mathbf{E}_1^3$ of a surface $M$ is said non-degenerated if the induced metric $x^*(\langle,\rangle)$ on $M$ is non-degenerate. In this setting, there is only two possibilities: if $x^*(\langle,\rangle)$ is positive definite , that is, it is a Riemmannian metric and the immersion is called *spacelike* or $x^*(\langle,\rangle)$ is a Lorentzian metric, that is, a metric of index 1, and the immersion is called *timelike*. For spacelike surfaces, the tangent planes are spacelike everywhere, and for timelike surfaces, they are timelike.

We consider spacelike or timelike surfaces in $\mathbf{E}_1^3$ that satisfy the relation

$$aH + bK = c, \tag{1}$$

where $H$ and $K$ are the mean curvature and the Gauss curvature of the surface, and $a$, $b$ and $c$ are constants. We say that the surface is a *linear Weingarten surface* of $\mathbf{E}_1^3$. In general, a Weingarten surface is a surface that satisfies a certain smooth relation $W = W(H, K) = 0$ and our case, that is, surfaces that satisfy (1) is the simplest case of $W$, that is, that $W$ is a linear function in its variables. The family of linear Weingarten surfaces include the surfaces with constant mean curvature ($b = 0$) and the surfaces with constant Gauss curvature ($a = 0$).

In this work we study linear Weingarten surfaces that are rotational, that is, invariant by a group of motions of $\mathbf{E}_1^3$ that pointwised fixed a straight-line. In such case, Equation (1) is a second ordinary differential equation that describes the shape of the generating curve of the surface. One can not expect to integrate this equation, because even in the trivial cases that $a = 0$ or $b = 0$, this integration is not possible. We are going to discard the cases that $H$ is constant of $K$ is constant, which are known: see for example [1, 2, 3]. We will obtain a first integration of (1). For the particular case that $a^2 - 4bc\epsilon = 0$, we describe all solutions, exactly, we have

**Theorem 1.1** *Let $M$ be a non-degenerate rotational surface in $\mathbf{E}_1^3$, and take $\epsilon = 1$ if $M$ is spacelike and $\epsilon = -1$ if $M$ is timelike. Assume that $M$ is a linear Weingarten surface*

*such that $a^2-4bc\epsilon=0$. After a rigid motion of the ambient space, a parametrization $X(u,v)$ of $M$ is as follows:*

1. *If the axis if timelike, $X(u,v)=(u\cos(v),u\sin(v),z(u))$, where*

$$z(u)=\pm\sqrt{\frac{4\epsilon b^2}{a^2}+(\frac{C}{a}\pm u)^2}+\mu,\quad C=2\sqrt{b\epsilon(-b+\lambda)},\ \mu,\lambda\in\mathbb{R}.$$

2. *If the axis is spacelike, we have two possibilities:*

   (a) *The parametrization is $X(u,v)=(u,z(u)\sinh(v),z(u)\cosh(v))$, where*

$$z(u)=\pm\frac{C}{a}\pm\sqrt{\frac{4\epsilon b^2}{a^2}\pm(u\pm\mu)^2},\quad C=2\sqrt{b\epsilon\lambda},\ \mu,\lambda\in\mathbb{R}.$$

   (b) *The parametrization is $X(u,v)=(u,z(u)\cosh(v),z(u)\sinh(v))$, where*

$$z(u)=\frac{-C}{a}\pm\sqrt{\frac{4b^2}{a^2}\pm(u\pm\mu)^2},\quad C=2\sqrt{b\lambda},\ \mu,\lambda\in\mathbb{R}.$$

3. *If the axis is lightlike, $X(u,v)=(-2uv,z(u)+u-uv^2,z(u)-u-uv^2)$, where*

$$z(u)=\frac{1}{48}\Big(\frac{-4ac\lambda+(cC^2-2a^2\lambda)u}{\epsilon c\lambda(2\lambda+cu^2)}+\epsilon\frac{cC^2+2a^2\lambda}{\sqrt{-2c\lambda}}\operatorname{arc\,tanh}(\sqrt{-\frac{c}{2\lambda}}\,u)\Big)+\mu,\quad \mu,\lambda\in\mathbb{R}.$$

## 2 Rotational surfaces in $\mathbf{E}_1^3$

In this section we describe the surfaces of revolution of $\mathbf{E}_1^3$ and we recall the concepts of mean curvature and Gauss curvature for a non-degenerate surface. We consider the rigid motions of the ambient space that leave a straight-line pointwised fixed, called, the axis of the surface. Let $L$ be the axis of the surface. Depending on $L$, there are three types of rotational motions. After an isometry of $\mathbf{E}_1^3$, the expressions of rotational motions with respect to the canonical basis $\{e_1,e_2,e_3\}$ are as follows:

$$R_v:\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\longmapsto\begin{pmatrix}\cos v&\sin v&0\\-\sin v&\cos v&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

$$R_v:\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\longmapsto\begin{pmatrix}1&0&0\\0&\cosh v&\sinh v\\0&\sinh v&\cosh v\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

$$R_v : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

See [4, 5] for more details.

**Definition 2.1** *A surface $M$ in $\mathbf{E}_1^3$ is a surface of revolution, or rotational surface, if $M$ is invariant by some of the above three groups of rigid motions.*

In particular, there exists a planar curve $\alpha = \alpha(u)$ that generates the surface, that is, $M$ is the set of points given by $\{R_v(\alpha(u)); u \in I, v \in \mathbb{R}\}$. We now describe the parametrizations of a rotational surface.

1. *Case $L$ is a timelike axis.* Consider that $L$ is the $x_3$-axis. If $p = (x_0, y_0, z_0) \notin L$, then $\{R_v(p); v \in \mathbb{R}\}$ is an Euclidean circle of radius $\sqrt{x_0^2 + y_0^2}$ in the plane $z = z_0$. If $\alpha(u) = (u, 0, z(u))$ is a planar curve in the plane $y = 0$, then the surface of revolution generated by $\alpha$ writes as
$$X(u,v) = (u\cos(v), u\sin(v), z(u)),\ \ u \neq 0. \tag{2}$$

2. *Case $L$ is a spacelike axis.* Consider that $L$ is the $x_1$-axis. If $p = (x_0, y_0, z_0)$ does not belong to $L$, then $\{R_v(p); v \in \mathbb{R}\}$ is an Euclidean hyperbola in the plane $x = x_0$ and with equation $y^2 - z^2 = y_0^2 - z_0^2$. For this kind of rotational surfaces, we have two type of surfaces:
   (a) If $\alpha(u) = (u, 0, z(u))$ is a planar curve in the plane $y = 0$, then the surface of revolution generated by $\alpha$ writes as
   $$X(u,v) = (u, z(u)\sinh(v), z(u)\cosh(v)),\ \ u \neq 0. \tag{3}$$
   (b) If $\alpha(u) = (u, z(u), 0)$ is a planar curve in the plane $z = 0$, then the surface is given by
   $$X(u,v) = (u, z(u)\cosh(v), z(u)\sinh(v)),\ \ u \neq 0. \tag{4}$$

3. *Case $L$ is a lightlike axis.* Consider that $L$ is the straight-line $v_1 =< (0,1,1) >$. If $p = (x_0, y_0, z_0)$ does not belong to the plane $< e_1, v_1 >$, the orbit $\{R_v(p); v \in \mathbb{R}\}$ is the curve
$$\beta(v) = (x - (y-z)v, xv + y - (y-z)\frac{v^2}{2}, xv + z - (y-z)\frac{v^2}{2}).$$
The curve $\beta$ lies in the plane $y - z = y_0 - z_0$ and describes a parabola in this plane, namely,
$$\beta(v) = (x,y,z) + v(-(y-z)e_1 + xv_1) - \frac{y-z}{2}v^2 v_1.$$

Consider $\alpha(u)$ a planar curve in the plane $<(0,1,1),(0,1,-1)>$ given as a graph on the straight-line $<(0,1,-1)>$, that is, $\alpha(u)=(0,u+z(u),-u+z(u))$. The surface of revolution generated by $\alpha$ is

$$X(u,v)=(-2uv, z(u)+u-uv^2, z(u)-u-uv^2),\ u\neq 0. \tag{5}$$

Let $M$ be surface and $x:M\rightarrow \mathbf{E}^3_1$ a non-degenerate immersion and we simply say that $M$ is non-degenerate. The surface could be not orientable, but if the immersion is spacelike, then $M$ is necessarily orientable. This is due to the following fact. At each point $p\in M$ there is two possible choices of a unit normal vector to the tangent plane $T_pM$ of $M$ at $p$. The normal vector to $M$ is a timelike vector, and in Minkowski space, two any timelike vectors are not orthogonal. Thus, if $E_3=(0,0,1)$, at each point $p\in M$, we take that unit normal vector $N(p)$ such that $\langle N(p),E_3\rangle<0$. This allows to define an global orientation on $M$, proving that $M$ is orientable. With this choice of $N$, we say that $N$ is future directed. In the case that the immersion is timelike, we will assume that $M$ is orientable.

Let $x:M\rightarrow\mathbf{E}^3_1$ be a non-degenerate immersion of a surface $M$ and let $N$ be a Gauss map. Let $U,V$ be vector fields to $M$ and we denote by $\nabla^0$ and $\nabla$ the Levi-Civitta connections of $\mathbf{E}^3_1$ and $M$ respectively. The Gauss formula says $\nabla^0_U V=\nabla_U V+\mathrm{II}(U,V)$, where II is the second fundamental form of the immersion. The Weingarten endomorphism is $A_p: T_pM\rightarrow T_pM$ defined as $A_p(U)=-(\nabla^0_U N)^\top_p=(-dN)_p(U)$. We have then $\mathrm{II}(U,V)=-\epsilon\langle\mathrm{II}(U,V),N\rangle N=-\epsilon\langle AU,V\rangle N$, where $\epsilon=1$ if $M$ is spacelike and $\epsilon=-1$ if $M$ is timelike. The mean curvature vector $\vec{H}$ is defined as $\vec{H}=(1/2)\mathrm{trace}(\mathrm{II})$ and the Gauss curvature $K$ as the determinant of II computed in both cases with respect to an orthonomal basis. The mean curvature $H$ is the function given by $\vec{H}=HN$, that is, $H=-\epsilon\langle\vec{H},N\rangle$. If $\{e_1,e_2\}$ is an orthonormal basis at each tangent plane, with $\langle e_1,e_1\rangle=1$, $\langle e_2,e_2\rangle=\epsilon$, then

$$\begin{aligned}\vec{H} &= \frac{1}{2}(\mathrm{II}(e_1,e_1)+\mathrm{II}(e_2,e_2))=-\epsilon\frac{1}{2}(\langle Ae_1,e_1\rangle+\epsilon\langle Ae_2,e_2\rangle)N=-\epsilon(\frac{1}{2}\mathrm{trace}(A))N\\ K &= -\epsilon\det(A).\end{aligned}$$

In this work we need to compute $H$ and $K$ using a parametrization of the surface. Let $X:D\subset\mathbb{R}^2\rightarrow\mathbf{E}^3_1$ be a parametrization of the surface, $X=X(u,v)$. Then $A=\mathrm{II}(\mathrm{I})^{-1}$, $\mathrm{I}=\langle,\rangle$ and we have the known formulae ([5]):

$$H=-\epsilon\frac{1}{2}\frac{eG-2fF+gE}{EG-F^2},\qquad K=-\epsilon\frac{eg-f^2}{EG-F^2}, \tag{6}$$

where $\{E,F,G\}$ and $\{e,f,g\}$ are the coefficients of I and II, respectively:

$$E=\langle X_u,X_u\rangle,\ F=\langle X_u,X_v\rangle,\ G=\langle X_v,x_v\rangle,$$

$$e=-\langle N_u,X_u\rangle,\ f=-\langle N_u,X_v\rangle,\ g=-\langle N_v,X_v\rangle,$$

where the subscripts denote the corresponding derivatives. Here $N$ is

$$N = \frac{X_u \times X_v}{\sqrt{\epsilon(EG - F^2)}}.$$

We recall that

$$W := EG - F^2 = \epsilon |X_u \times X_v|^2 \left\{ \begin{array}{l} \text{is positive if } M \text{ is spacelike} \\ \text{is negative if } M \text{ is timelike} \end{array} \right.$$

Finally, in order to the computations for $H$ and $K$, we recall that the cross-product $\times$ satisfies that for any vectors $u, v, w \in \mathbf{E}_1^3$, $\langle u \times v, w\rangle = \det(u, v, w)$. Then (6) writes as

$$H = -\frac{\epsilon}{2}\frac{G\det(X_u, X_v, X_{uu}) - 2F\det(X_u, X_v, X_{uv}) + E\det(X_u, X_v, X_{vv})}{(\epsilon(EG - F^2))^{3/2}} := \frac{H_1}{2W^{3/2}}. \quad (7)$$

$$K = -\frac{\det(X_u, X_v, X_{uu})\det(X_u, X_v, X_{vv}) - \det(X_u, X_v, X_{uv})^2}{(EG - F^2)^2} := \frac{K_1}{W^2}. \quad (8)$$

In Minkowski ambient space, the role of spheres is played by pseudohyperbolic surfaces and pseudospheres [4]. If $p_0 \in \mathbf{E}_1^3$ and $r > 0$ the pseudohyperbolic surface centered at $p_0$ with radius $r > 0$ is $\mathbf{H}^{2,1}(r; p_0) = \{p \in \mathbf{E}_1^3; \langle p - p_0, p - p_0\rangle = -r^2\}$ and the pseudosphere centered at $p_0$ and radius $r > 0$ is $\mathbf{S}^{2,1}(r; p_0) = \{p \in \mathbf{E}_1^3; \langle p - p_0, p - p_0\rangle = r^2\}$. If $M$ is spacelike (resp. timelike) then $N$ is timelike (resp. spacelike) and $N : M \to \mathbf{H}^{2,1}(1)$ (resp. $N : M \to \mathbf{S}^{2,1}(1)$), where $\mathbf{H}^{2,1}(1) = \mathbf{H}^{2,1}(1; O)$ (resp. $\mathbf{S}^{2,1}(1) = \mathbf{S}^{2,1}(r; O)$, being $O$ the origin of coordinates of $\mathbb{R}^3$. For both kind of surfaces, we can take $N(p) = (p - p_0)/r$ and $A = -\frac{1}{r}I$. Then $H = \epsilon/r$ and $K = -\epsilon/r^2$.

# 3 Rotational surfaces with timelike axis

We assume that the generating curve $\alpha$ lies in the $xz$-plane and we parametrize $\alpha$ as the graph of a function $z = z(u)$, that is, $\alpha(u) = (u, 0, z(u))$, $u > 0$. Then the surface is parametrized as in (2) and $W = u^2(1 - z^2)$. Thus $z'^2 < 1$ if the surface is spacelike and $z'^2 > 1$ if $M$ is timelike. Using (7) and (8), the expressions of $H$ and $K$ are:

$$H = -\frac{1}{2}\left(\frac{\epsilon z'}{u\sqrt{\epsilon(1 - z'^2)}} + \frac{z''}{(\epsilon(1 - z'^2))^{3/2}}\right), \quad K = -\frac{z' z''}{u(1 - z'^2)^2}.$$

Then the relation (1) writes as

$$\frac{a}{2}\left(\frac{\epsilon z'}{u\sqrt{\epsilon(1 - z'^2)}} + \frac{z''}{(\epsilon(1 - z'^2))^{3/2}}\right) + b\frac{z' z''}{u(1 - z'^2)^2} = -c.$$

Multiplying by $u$ we obtain a first integral. Exactly, we have

$$a\left(u\frac{\epsilon z'}{\sqrt{\epsilon(1-z'^2)}}\right)'+b\left(\frac{1}{1-z'^2}\right)'=-2cu.$$

Then there exists a integration constant $\lambda\in\mathbb{R}$ such that

$$\epsilon\frac{auz'}{\sqrt{\epsilon(1-z'^2)}}+\frac{b}{1-z'^2}=-cu^2+\lambda. \tag{9}$$

Let

$$\phi=\frac{z'}{\sqrt{\epsilon(1-z'^2)}}.$$

Then $1+\epsilon\phi^2=1/(1-z'^2)$ and Equation (9) writes as $b\phi^2+au\phi+\epsilon(b+cu^2-\lambda)=0$. Hence, we obtain $\phi$:

$$\frac{z'}{\sqrt{\epsilon(1-z'^2)}}=\frac{-au\pm\sqrt{(a^2-4bc\epsilon)u^2+4b\epsilon(-b+\lambda)}}{2b}. \tag{10}$$

We completely solve this differential equation in two particular cases:

1. Consider $\lambda=b$. Then we have

   $$\frac{z'}{\sqrt{\epsilon(1-z'^2)}}=\frac{-a\pm\sqrt{a^2-4bc\epsilon}}{2b}u=Cu,\quad C=\frac{-a\pm\sqrt{a^2-4bc\epsilon}}{2b}.$$

   Then

   $$z(u)=\pm\frac{\sqrt{\epsilon+C^2u^2}}{C}+\mu,\ \ \mu\in\mathbb{R}.$$

   From the parametrization (2) of the surface, one concludes that $M$ satisfies the equation $x^2+y^2-(z-\mu)^2=-\frac{\epsilon}{C^2}$. Letting $p_0=(0,0,\mu)$, if $\epsilon=1$, the surface $M$ is the pseudohyperbolic surface $\mathbf{H}^{2,1}(1/|C|;p_0)$ and when $\epsilon=-1$, $M$ is a pseudosphere $\mathbf{S}^{2,1}(1/|C|;p_0)$.

2. Assume $a^2-4bc\epsilon=0$. Then

   $$\frac{z'}{\sqrt{\epsilon(1-z'^2)}}=\frac{-au\pm C}{2b},\quad C=2\sqrt{b\epsilon(-b+\lambda)}.$$

   The integration of this equation is

   $$z(u)=\pm\sqrt{\frac{4\epsilon b^2}{a^2}+(\frac{C}{a}\pm u)^2}+\mu,\ \ \mu\in\mathbb{R}.$$

## 4 Rotational surfaces with spacelike axis

We distinguish two cases according the two possible parametrizations.

1. Case I. Assume that the parametrization is given by (3). The relation (1) writes as

$$\frac{a}{2}\left(\frac{\epsilon}{z\sqrt{\epsilon(1-z'^2)}}+\frac{z''}{(\epsilon(1-z'^2))^{3/2}}\right)+b\frac{z''}{z(1-z'^2)^2}=-c.$$

   Multiplying by $zz'$, we obtain a first integral. Exactly, we have

$$a\left(\frac{\epsilon z}{\sqrt{\epsilon(1-z'^2)}}\right)'+b\left(\frac{1}{1-z'^2}\right)'=-c(z^2)'.$$

   Then there exists an integration constant $\lambda\in\mathbb{R}$ such that

$$\epsilon\frac{az}{\sqrt{\epsilon(1-z'^2)}}+\frac{b}{1-z'^2}=-cz^2+\lambda. \tag{11}$$

   Now we take $\phi=1/\sqrt{\epsilon(1-z'^2)}$. Then Equation (11) writes as

$$b\phi^2+az\phi+\epsilon(cz^2-\lambda)=0.$$

   Then

$$\frac{1}{\sqrt{\epsilon(1-z'^2)}}=\frac{-az\pm\sqrt{(a^2-4bc\epsilon)z^2+4b\epsilon\lambda}}{2b}. \tag{12}$$

   We completely solve this differential equation in two particular cases:

   (a) Consider $\lambda=0$. Then we have

$$\frac{1}{\sqrt{\epsilon(1-z'^2)}}=\frac{-a\pm\sqrt{a^2-4bc\epsilon}}{2b}z=Cz,\quad C=\frac{-a\pm\sqrt{a^2-4bc\epsilon}}{2b}.$$

   The solution of this differential equation is

$$z(u)=\pm\sqrt{\frac{\epsilon}{C^2}\pm(u\pm C\mu)^2},\ \mu\in\mathbb{R}\}.$$

   From the parametrization (3) of the surface, one concludes that $M$ satisfies the equation $(x-C\mu)^2+y^2-z^2=-\frac{\epsilon}{C^2}$. Thus, if we set $p_0=(\pm C\mu,0,0)$, for $\epsilon=1$ we obtain that $M$ is the pseudohyperbolic surface $\mathbf{H}^{2,1}(1/|C|;p_0)$ and for $\epsilon=-1$, $M$ is the pseudosphere $\mathbf{S}^{2,1}(1/|C|;p_0)$.

(b) Assume $a^2 - 4bc\epsilon = 0$. Then

$$\frac{1}{\sqrt{\epsilon(1-z'^2)}} = \frac{-az \pm C}{2b}, \quad C = 2\sqrt{b\epsilon\lambda}.$$

The integration of this equation is

$$z(u) = \pm\frac{C}{a} \pm \sqrt{\frac{4\epsilon b^2}{a^2} \pm (u \pm \mu)^2}, \ \mu \in \mathbb{R}.$$

2. Case II. The expression of the parametrization is written in (4). In this case, the surface is timelike, since $EG - F^2 = -z^2(1+z'^2)$. The Weingarten relation (1) is

$$\frac{a}{2}\left(\frac{-1}{z\sqrt{1+z'^2}} + \frac{z''}{(1+z'^2)^{3/2}}\right) - b\frac{z''}{z(1+z'^2)^2} = c.$$

Multiplying by $zz'$ again, we have

$$-a\left(\frac{z}{\sqrt{1+z'^2}}\right)' + b\left(\frac{1}{1+z'^2}\right)' = c(z^2)'.$$

It follows the existence of an integration constant $\lambda \in \mathbb{R}$ such that

$$-\frac{az}{\sqrt{1+z'^2}} + \frac{b}{1+z'^2} = cz^2 + \lambda. \tag{13}$$

If we set $\phi = 1/\sqrt{1+z'^2}$, Equation (13) is $b\phi^2 - az\phi - cz^2 - \lambda = 0$, obtaining

$$\frac{1}{1+z'^2} = \frac{az \pm \sqrt{(a^2+4bc)z^2 + 4b\lambda}}{2b}. \tag{14}$$

As in the previous case, we solve this equation in the next two cases:

(a) If $\lambda = 0$, then

$$\frac{1}{\sqrt{1+z'^2}} = \frac{-a \pm \sqrt{a^2+4bc}}{2b}z = Cz, \quad C = \frac{a \pm \sqrt{a^2+4bc}}{2b}.$$

The solution of this equation is

$$z(u) = \pm\sqrt{\frac{1}{C^2} - (u \pm C\mu)^2}, \ \mu \in \mathbb{R}\}.$$

This surface is the pseudosphere $\mathbf{S}^{2,1}(1/|C|; p_0)$, with $p_0 = (\pm C\mu, 0, 0)$ since by the expression of the parametrization (4), the coordinates of $M$ satisfies $(x \pm C\mu)^2 + y^2 - z^2 = 1/C^2$.

(b) If $a^2+4bc=0$, then

$$\frac{1}{\sqrt{1+z'^2}}=\frac{az\pm C}{2b},\quad C=2\sqrt{b\lambda}.$$

The solution of this equation is

$$z(u)=\frac{-C}{a}\pm\sqrt{\frac{4b^2}{a^2}\pm(u\pm\mu)^2},\ \mu\in\mathbb{R}.$$

# 5 Rotational surfaces with lightlike axis

Consider the parametrization given in (5). Then $EG-F^2=16u^2z'$ and the relation (1) writes as

$$\frac{a}{2}\left(\frac{1}{2u\sqrt{\epsilon z'}}-\frac{\epsilon z''}{4(\epsilon z')^{3/2}}\right)+b\frac{z''}{8uz'^2}=c.$$

Multiplying by $u$ we obtain a first integral. Exactly, we have

$$\frac{a}{4}\left(\frac{u}{\sqrt{\epsilon z'}}\right)'-\frac{b}{8}\left(\frac{1}{z'}\right)'=cu.$$

Then there exists a integration constant $\lambda\in\mathbb{R}$ such that

$$\frac{a}{4}\frac{u}{\sqrt{\epsilon z'}}-\frac{b}{8z'}=\frac{c}{2}u^2+\lambda. \tag{15}$$

From (15), we obtain the value of $\sqrt{\epsilon z'}$:

$$\sqrt{\epsilon z'}=\frac{a\epsilon u\pm\sqrt{(a^2-4bc\epsilon)u^2-8b\epsilon\lambda}}{4\epsilon(cu^2+2\lambda)}.$$

As in the two previous cases, we distinguish two special cases:

1. If $\lambda=0$, then

$$\sqrt{\epsilon z'}=\frac{a\pm\epsilon\sqrt{a^2-4bc\epsilon}}{4c}\frac{1}{u}:=\frac{C}{u},\quad C=\frac{a\pm\epsilon\sqrt{a^2-4bc\epsilon}}{4c}.$$

We solve this equation obtaining

$$z(u)=-\frac{\epsilon C^2}{u}+\mu,\quad \mu\in\mathbb{R}.$$

From the parametrization (5), we see that $M$ satisfies the equation $x^2+y^2-(z-\mu)^2=-4\epsilon C^2$. Thus, if $p_0=(0,0,\mu)$, we have that $M=\mathbf{H}^{2,1}(2|C|;p_0)$ if $\epsilon=1$, and $M=\mathbf{S}^{2,1}(2|C|;p_0)$ if $\epsilon=-1$.

2. Assume $a^2 - 4bc\epsilon = 0$. Then

$$\sqrt{\epsilon z'} = \frac{a\epsilon u \pm C}{4\epsilon(cu^2 + 2\lambda)}, \quad C = \sqrt{-8b\epsilon\lambda}.$$

We point out that $-8b\epsilon\lambda > 0$ and that combining with $a^2 - 4bc\epsilon = 0$, we have $c\lambda \leq 0$. The solution is

$$z(u) = \frac{1}{64}\Big(\frac{\mp 4aC\lambda \pm \epsilon(cC^2 - 2a^2\lambda)u}{\epsilon c\lambda(2\lambda + cu^2)} + \epsilon\frac{cC^2 + 2a^2\lambda}{\sqrt{-2c^3\lambda^3}}\operatorname{arc\,tanh}(\sqrt{-\frac{c}{2\lambda}}\, u)\Big) + \mu, \quad \mu, \lambda \in \mathbb{R}.$$

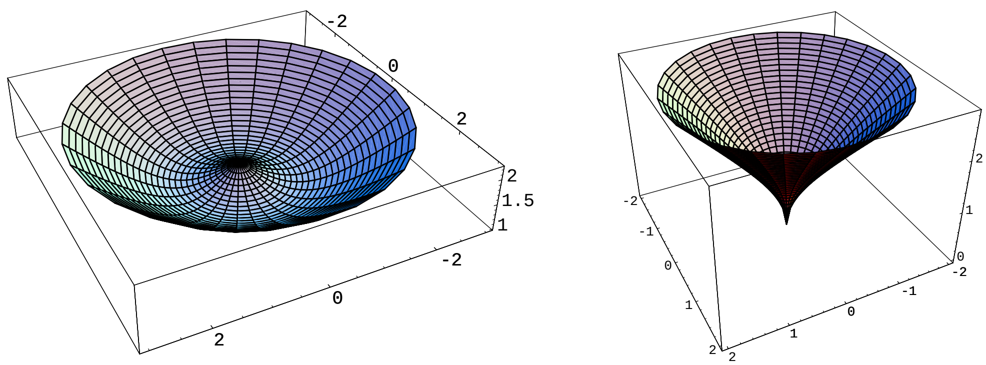


Figure 1: Rotational surfaces with timelike axis, for $a = 2$, $b = \epsilon$ and $\mu = 0$: The surface is spacelike with $\lambda = 2$ (left). The surface is timelike with $\lambda = 0$ (right).

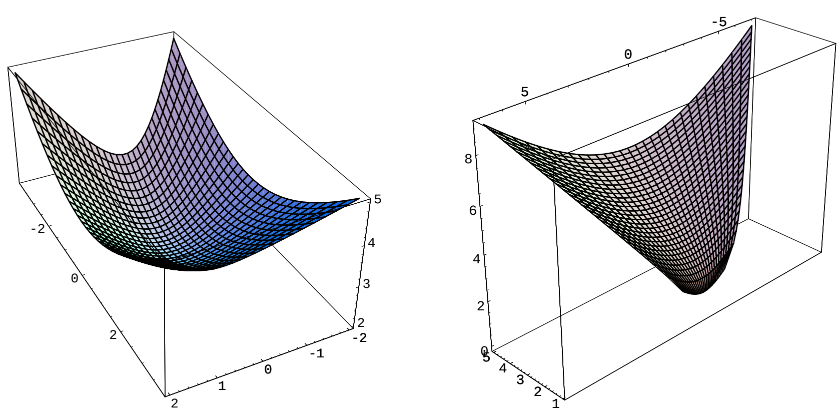


Figure 2: Rotational surfaces with spacelike axis, for $a = 2$, $b = \epsilon$, $\lambda = 1$ and $\mu = 0$: The surface is spacelike (left). The surface is timelike (right).

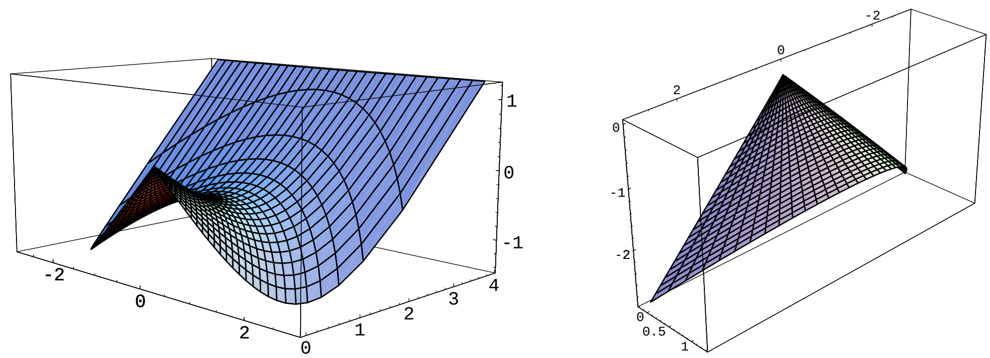


Figure 3: Rotational surfaces with lightlike axis, for $a = 2$, $b = -\epsilon$, $\lambda = 1$ and $\mu = 0$: The surface is spacelike (left). The surface is timelike (right).

## References


[1] J.I. Hano, K. Nomizu, Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, Tohoku Math. J., 36 (1984), 427-435.

[2] R. López, Timelike surfaces in Lorentz 3-space with constant mean curvature, Tohoku Math. J., 52 (2000), 515–532.

[3] R. López, Surfaces of constant Gauss curvature in Lorentz-Minkowski 3-space, Rocky Mount. J. Math., 33 (2003), 971–993.

[4] B. ONeill, Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1983.

[5] T. Weinstein, An introduction to Lorentz surfaces, Walter de Gruyter, Berlin, 1995.